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Stationary Natural Frequencies and Mode Shapes of Composite Laminates

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The natural vibration of composite laminates is studied in this article. It is found that for a composite laminate of constant thickness the natural frequency approaches monotonically to a stationary value, and the natural vibration mode a stationary shape, as the number of its constituent plies increases. The stationary value and the stationary shape can easily be determined by utilizing the effective modulus and the effective density of the composite laminate, for which exact expressions are derived. The effective modulus in natural vibrations equals that in static problems in all cases, whereas the effective density, depending on frequency and other parameters in general, coincides with the mean density of the constituent plies only in simple vibration modes.

Introduction

STATIC analysis of composite materials has extensively been treated in many references; see Refs. 1–3, for instance. On the other hand, behavior of composite materials in dynamic problems does not appear to have been studied sufficiently. Sun et al.[4] developed a continuum theory for laminated media, establishing dispersion relations in wave propagation. Chang[5] treated some natural vibration problems for composite laminates, where the feasibility of the effective modulus approach in dynamic problems of composite materials was also discussed. In a recent paper,[6] Lamb wave propagation in anisotropic laminates was investigated, with dispersion relation and energy distribution being the focuses of the study. However, although some complicated problems have been touched on or discussed, for a number of basic dynamic problems of composite laminates, such as those that will be treated in this article, accurate computation and incisive physical insight are still insufficient or inconclusive, and further significant investigations are needed.

Composite laminates are composed of various constituent plies. When an exact solution to the natural vibration of an isolated ply is known, that of the composite laminate can usually be found, without much difficulty, by a suitable combining of solutions for individual plies in case the ply number of the laminate is small. However, for large ply numbers, the combining process becomes cumbersome, and therefore the alternative approach of effective modulus may be invoked. Consequently, in studying the natural vibration behavior of composite laminates, emphasis should be laid on those that consist of a large number of constituent plies.

This article discusses the natural vibration behavior of composite laminates. Exact solutions to the problem based on the theory of elasticity are first provided. To investigate the problem in the case of large ply numbers, expressions for the effective modulus and the effective density in natural vibrations are derived for composite laminates, with which solutions of good accuracy for large ply numbers can easily be obtained. It is found that the effective modulus equals that for static problems in all cases, whereas the effective density, depending in general also on frequency and other parameters, is not a material constant in the conventional sense and coincides with the mean density of the constituent plies only in simple vibration modes. Therefore, in dynamic problems of composite laminates, this effective density should be understood as a generalized material constant.

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Longitudinal (Through-the-Thickness) Vibration

A composite laminate of constant thickness H and composed of P constituent plies is depicted and described in Fig. 1, where notations h_i, λ_i, μ_i, and c_i ($i = 1, 2, \ldots, P$) are used to denote the thickness, Lame constants, and the longitudinal wave speed of the ith ply, respectively. Also, a z_i axis is fixed to the ply as shown in the figure to simplify the formulation.

Now consider the natural longitudinal vibration of the composite laminate. Assuming that the composite laminate is fully restrained in the x_i and y_i directions, so that the motion is one dimensional, we write the expressions for the displacement and normal stress in the thickness direction for the plies as follows[7]:

$$\bar{u}_i(z_i, t) = u_i(z_i) \exp(\sqrt{-1}\,\omega t) \tag{1}$$

$$\bar{\sigma}_i(z_i, t) = D_i \frac{\partial u_i}{\partial z_i} \exp(\sqrt{-1}\,\omega t) \tag{2}$$

where ω stands for the natural frequency and $D_i = \lambda_i + 2\mu_i$. The basic unknown, $u_i(z_i)$, should be sought from the following equations:

$$\frac{d^2 u_i}{dz_i^2} + \beta_i^2 u_i = 0, \qquad i = 1, 2, \ldots, P \tag{3}$$

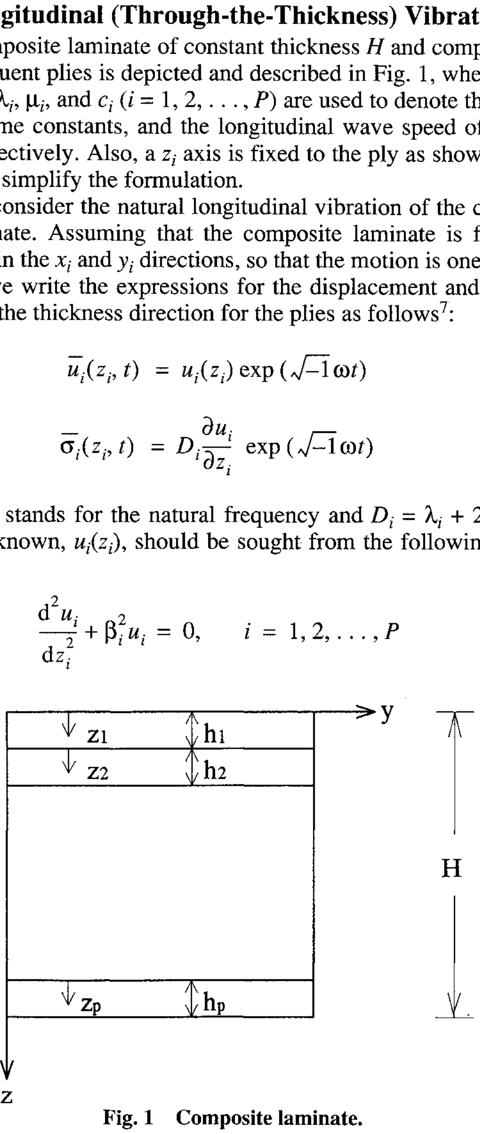

Fig. 1 Composite laminate.

where $\beta_i = \omega/c_i$. The solutions to the previous equations are[7]

$$u_i(z_i) = A_i \sin(\beta_i z_i) + B_i \cos(\beta_i z_i), \qquad i = 1, 2, \ldots, P \tag{4}$$

with

$$A_i = \frac{D_{i-1}\beta_{i-1}}{D_i \beta_i}[A_{i-1}\cos(\beta_{i-1}h_{i-1}) - B_{i-1}\sin(\beta_{i-1}h_{i-1})] \tag{5}$$

$$B_i = A_{i-1}\sin(\beta_{i-1}h_{i-1}) + B_{i-1}\cos(\beta_{i-1}h_{i-1}) \tag{6}$$

because at the interfaces $z_i = 0$, $i = 2, \ldots, P$, the continuity of displacement and stress holds.

In Eq. (4), $A_1 = 0$, since the upper surface of the laminate, $z_1 = 0$, is traction free; B_1 is selected as an amplitude parameter and therefore can be of an arbitrary nonvanishing value. The remaining unknown coefficients A_i, B_i ($i = 2, \ldots, P$), and ω can be determined from Eqs. (5) and (6) and the traction-free boundary condition of the lower surface of the laminate, $z_p = h_p$:

$$A_p \cos(\beta_p h_p) - B_p \sin(\beta_p h_p) = 0 \tag{7}$$

All of the solutions, Eqs. (4–7), are exact formulas of the theory of elasticity. Values of the natural frequency ω can exactly be evaluated from the solutions. The detailed numerical result of ω for a particular case ($H = 1$, $h_1 = h_2 = h_3 = \ldots = h_p = H/P$, $c_1 = c_3 = c_5 = \ldots = c_{p-1} = c$, $c_2 = c_4 = c_6 = \ldots = c_p = 2c$, $D_1 = D_3 = D_5 = \ldots = D_{p-1} = D$, $D_2 = D_4 = D_6 = \ldots = D_p = 3D/2$) is presented in Table 1. Once ω has been determined, the corresponding natural vibration mode can be found out by utilizing Eq. (4). However, for large ply number P, the solution procedure may become difficult to carry out since Eqs. (5–7) involve $2P - 1$ coupled transcendent equations. On the other hand, it is seen from Table 1 that as the ply number P increases endlessly, the natural frequency definitely and monotonically approaches a limit, the stationary value. This stationary value, in association with the corresponding stationary vibration mode to be discussed later on, can be used to approximate accurately the natural vibration behavior of a composite laminate of large ply number P and represents the ultimate state the composite laminate can reach during its natural vibration, when the thickness of its constituent plies approaches zero.

Table 1 Numerical values of first five natural frequency parameters ω_k/c for longitudinal vibration

	k[a]				
P[b]	1	2	3	4	5
2	1.3875	2.6128	4.0000	5.3873	6.6127
4	1.3183	2.7744	4.0000	5.2256	6.6812
6	1.3201	2.6288	4.1617	5.3533	6.6467
8	1.3206	2.6366	3.9316	5.5490	6.7130
10	1.3208	2.6390	3.9491	5.2272	6.9362
12	1.3209	2.6401	3.9549	5.2577	6.5160
14	1.3210	2.6408	3.9576	5.2680	6.5623
16	1.3210	2.6412	3.9592	5.2731	6.5786
18	1.3211	2.6414	3.9602	5.2762	6.5868
20	1.3211	2.6416	3.9609	5.2780	6.5914
22	1.3211	2.6417	3.9614	5.2794	6.5946
24	1.3211	2.6418	3.9617	5.2805	6.5967
26	1.3211	2.6419	3.9620	5.2811	6.5983
28	1.3211	2.6420	3.9622	5.2816	6.5997
30	1.3211	2.6420	3.9624	5.2821	6.6005
32	1.3212	2.6421	3.9625	5.2824	6.6011
34	1.3212	2.6421	3.9627	5.2829	6.6017
36	1.3212	2.6421	3.9627	5.2830	6.6022
38	1.3212	2.6422	3.9628	5.2831	6.6026
40	1.3212	2.6422	3.9629	5.2832	6.6030
42	1.3212	2.6422	3.9629	5.2834	6.6033
44	1.3212	2.6422	3.9630	5.2836	6.6036
46	1.3212	2.6422	3.9630	5.2838	6.6039
48	1.3212	2.6422	3.9631	5.2839	6.6042
50	1.3212	2.6422	3.9631	5.2840	6.6044
52	1.3212	2.6423	3.9631	5.2840	6.6046
54	1.3212	2.6423	3.9632	5.2842	6.6048
56	1.3212	2.6423	3.9632	5.2842	6.6049
58	1.3212	2.6423	3.9632	5.2842	6.6050
60	1.3212	2.6423	3.9633	5.2843	6.6050
SV[c]	1.3212	2.6423	3.9635	5.2846	6.6058

[a]Mode number. [b]Number of plies. [c]Stationary value.

In view of the preceding points and for the purpose of determining the stationary value of the natural frequency, the effective modulus approach is to be used to alternatively solve the problem. To this end, a composite laminate consisting of two constituent plies, called a two-ply basic unit, is first considered. It is easy to see that for this unit, any particular form of periodic motion is completely prescribed by the displacement and stress on its upper surface or by three constants A_1, B_1, and ω contained in Eqs. (4); A_2 and B_2, which characterize the motion of the second ply, depend on them and by Eqs. (5) and (6) can directly be determined from those three constants.

Now suppose there is an equivalent counterpart of the basic unit, that is, a homogeneous plate of the same thickness as that of the basic unit, $h_e = h_1 + h_2$, and with material constants c_e and $D_e = \lambda_e + 2\mu_e$. The displacement solution to the natural vibration of the homogeneous plate, u_e, is also given by Eqs. (4) with the subscript i replaced by e, $z = z_e$. It is attempted to fix the constants c_e and D_e based on the requirement that the natural vibration behavior of the composite basic unit and the homogeneous plate be identical to each other in the following sense: when a certain displacement and a certain stress are both assigned to the upper surfaces of the two systems, that is, when the following two equations hold,

$$B_1 = B_e \tag{8a}$$

$$D_1 \beta_1 A_1 = D_e \beta_e A_e \tag{8b}$$

the displacements and stresses on the lower surfaces of the two systems will be mutually equal; that is, the following equations should be true:

$$A_2 \sin(\beta_2 h_2) + B_2 \cos(\beta_2 h_2) = A_e \sin(\beta_e h_e) + B_e \cos(\beta_e h_e) \tag{9}$$

$$D_2\beta_2[A_2\cos(\beta_2 h_2) - B_2 \sin(\beta_2 h_2)] = D_e \beta_e [A_e \cos(\beta_e h_e) - B_e \sin(\beta_e h_e)] \tag{10}$$

It is found that c_e and D_e, obtained by solving Eqs. (8–10) in association with Eqs. (5) and (6), are variables depending on frequency ω. However, in case h_1, h_2, and h_e are small, so that $\beta_1 h_1$, $\beta_2 h_2$, and $\beta_e h_e \ll 1$ and quantities equal to or smaller than $\mathcal{O}(\beta_1^2 h_1^2)$, $\mathcal{O}(\beta_2^2 h_2^2)$, and $\mathcal{O}(\beta_e^2 h_e^2)$ can be neglected, the expressions for c_e and D_e become

$$c_e = \left[\frac{D_e(h_1 + h_2)}{D_1 h_1/c_1^2 + D_2 h_2/c_2^2}\right]^{1/2} \tag{11}$$

$$D_e = \frac{h_1 + h_2}{h_1/D_1 + h_2/D_2} \tag{12}$$

which are independent of ω. Furthermore, using the well-known relation $c = \sqrt{D/\rho}$, with ρ being the density of the material, the expression for the effective density of the basic unit, or the density of the homogeneous plate, can be obtained from Eq. (11) as follows:

$$\rho_e = \frac{\rho_1 h_1 + \rho_2 h_2}{h_1 + h_2} \tag{13}$$

These effective quantities depend solely on the material and geometric constants of the constituent plies. In fact, ρ_e is the mean density of the basic unit, and D_e is identical to that defined in static problems for composite laminates.[1]

Expressions for c_e and D_e can easily be extended to basic units consisting of more than two constituent plies. Since now the two-ply basic unit can be regarded as a homogeneous ply of thickness h_e and with material constants c_e, D_e, and ρ_e, the effective quantities of a three-ply basic unit can be written directly as

$$c_e(3) = \left[\frac{D_e(3)(h_1+h_2+h_3)}{D_1h_1/c_1^2+D_2h_2/c_2^2+D_3h_3/c_3^2}\right]^{1/2} \tag{14}$$

$$D_e(3) = \frac{h_1+h_2+h_3}{h_1/D_1+h_2/D_2+h_3/D_3} \tag{15}$$

$$\rho_e(3) = \frac{\rho_1h_1+\rho_2h_2+\rho_3h_3}{h_1+h_2+h_3} \tag{16}$$

Expressions for $c_e(N)$, $D_e(N)$, and $\rho_e(N)$, with the ply number N being an integer larger than 3, can be obtained by deduction:

$$c_e(N) = \left[\frac{D_e(N)(h_1+h_2+\cdots+h_N)}{D_1h_1/c_1^2+D_2h_2/c_2^2+\cdots+D_Nh_N/c_N^2}\right]^{1/2} \tag{17}$$

$$D_e(N) = \frac{h_1+h_2+\cdots+h_N}{h_1/D_1+h_2/D_2+\cdots+h_N/D_N} \tag{18}$$

$$\rho_e(N) = \frac{\rho_1h_1+\rho_2h_2+\cdots+\rho_Nh_N}{h_1+h_2+\cdots+h_N} \tag{19}$$

For a composite laminate composed of a number of N-ply basic units of the same composition, its effective quantities are identical to those of the N-ply basic unit expressed by Eqs. (17–19), since the composite laminate can be regarded as composed of some homogeneous plates of the same kind.

The natural frequency and vibration mode of a homogeneous plate are simply given by[7] the following when $j = 1, 2, 3, \ldots,$

$$u_e \doteq B_e \cos(\beta_e z) \tag{20a}$$

$$\beta_e = \frac{jc_e(N)\pi}{H} \tag{20b}$$

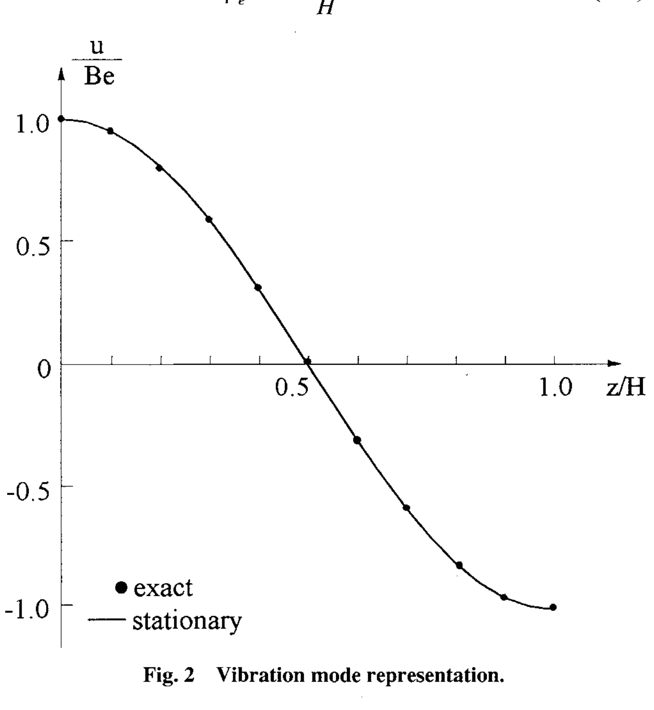

Fig. 2 Vibration mode representation.

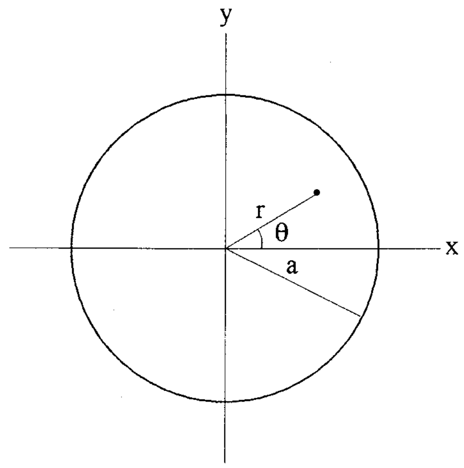

Fig. 3 Cross section of the laminated cylinder.

This natural frequency and vibration mode are good approximations to those of the corresponding composite laminate, provided the thickness of its basic units is small. In the limit case when the thickness approaches zero, they stand for true solution to the natural vibration of the composite laminate. In other words, the natural vibration behavior of the composite laminate is nearly identical to that of the homogeneous plate of the same thickness H and with material constants $c_e(N)$, $D_e(N)$, and $\rho_e(N)$ in case of thickness of the basic unit is small; moreover, the identity is exact when the thickness approaches zero.

It can be seen from the numerical example presented in Table 1 that for small ply numbers the natural frequency of the composite laminate, whose geometric and material constants have been indicated earlier, varies rather irregularly; however, from a certain ply number on, it definitely and monotonically approaches a stationary value as the ply number increases. The stationary value agrees perfectly with the natural frequency of the homogeneous plate evaluated from Eq. (20b). Meanwhile, the corresponding stationary mode can easily be determined from Eq. (20a).

Figure 2 shows the first mode shape of the composite laminate in the preceding example with $P = 20$ and that of the homogeneous plate or the stationary mode shape of the composite laminate. It is seen that no discrepancy can be found in the diagram between these two mode shapes.

Torsional (In-Plane) Vibration

In this section we consider the natural torsional vibration of a composite laminated cylinder of radius a and length H. A diametrical cross section of the cylinder is also represented by the diagram in Fig. 1, whereas Fig. 3 shows the cross section of the cylinder perpendicular to the z axis.

The time-independent portion of the unique nonvanishing displacements in the torsional vibration are $V_i(r, z)$, the displacements in the θ direction with the (r, θ) plane being perpendicular to the z axis. These displacements are governed by the following equation[7]:

$$\frac{\partial^2 V_i}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial V_i}{\partial r} - \frac{V_i}{r^2} + \frac{\partial^2 V_i}{\partial z_i^2} + \frac{\omega^2 V_i}{c_{si}^2} = 0, \qquad i = 1, 2, \ldots, P \tag{21}$$

where c_{si} denotes the shear wave speed of the ith ply material. The time-independent portion of the nonzero stresses are

$$\tau_{\theta z_i} = \mu_i \frac{\partial V_i}{\partial z_i} \tag{22a}$$

$$\tau_{r\theta_i} = \mu_i\left(\frac{\partial V_i}{\partial r} - \frac{V_i}{r}\right), \qquad i = 1, 2, \ldots, P \tag{22b}$$

It is shown[7] that the solutions $V_i(r, z)$ satisfying Eq. (21) take the following form:

$$V_i(r, z) = J_1(\beta r)\,[A_i \sin(\xi_i z_i) + B_i \cos(\xi_i z_i)], \quad i = 1, 2, \ldots, P \tag{23}$$

where $\xi_i = (\omega^2/c_{si}^2 - \beta^2)^{1/2}$, with β being the root of the following equation:

$$\beta a J_0(\beta a) - 2J_1(\beta a) = 0 \tag{24}$$

to satisfy the free boundary condition on the lateral surface:

$$r = a, \quad \tau_{r\theta_i} = 0, \quad i = 1, 2, \ldots, P \tag{25}$$

The traction-free boundary condition on the upper surface of the cylinder requires $A_1 = 0$. As indicated in the preceding section, B_1 can be an arbitrary nonzero constant. In addition, the free boundary condition of the lower surface and the continuity conditions of displacement and stress at the interfaces $z_i = 0$, $i = 2, 3, \ldots, P$, provide the following equations to determine ω, A_i, and B_i, $i = 2, 3, \ldots, P$:

$$A_i = \frac{\mu_{i-1}\xi_{i-1}}{\mu_i \xi_i}[A_{i-1}\cos(\xi_{i-1}h_{i-1}) - B_{i-1}\sin(\xi_{i-1}h_{i-1})] \quad i = 2, 3, \ldots, P \tag{26}$$

$$B_i = A_{i-1}\sin(\xi_{i-1}h_{i-1}) + B_{i-1}\cos(\xi_{i-1}h_{i-1}) \quad i = 2, 3, \ldots, P \tag{27}$$

$$A_P \cos(\xi_P h_P) - B_P \sin(\xi_P h_P) = 0 \tag{28}$$

For the purpose of determining the effective quantities in this case, we again consider a composite laminate in the form of a two-ply basic unit and a corresponding homogeneous circular plate of radius a, thickness $h_e = h_1 + h_2$, and material constants μ_e, ρ_e, and c_{se}. The natural vibration mode and frequency of the homogeneous plate, for the case $\beta = 0$, the first root of Eq. (24), are simply given by[7]

$$V_e = B_e r \cos(\xi_e z) \tag{29a}$$

$$\omega = \frac{m c_{se}\pi}{H} \quad m = 1, 2, 3, \ldots \tag{29b}$$

Those in association with other roots of Eq. (24) are expressed by

$$V_e = B_e J_1(\beta r)\cos(\xi_e z) \tag{30a}$$

$$\omega = c_{se}\sqrt{\frac{m^2\pi^2}{H^2} + \beta^2} \quad m = 1, 2, 3, \ldots \tag{30b}$$

We now proceed to determine the effective quantities of the composite cylinder following a line similar to that applied in the preceding section. In this way, the following equations ensue:

$$B_1 = B_e \tag{31a}$$

$$\mu_1 \xi_1 A_1 = \mu_e \xi_e A_e \tag{31b}$$

$$\mu_1 \xi_1^2 h_1 + \mu_2 \xi_2^2 h_2 = \mu_e \xi_e^2 h_e \tag{32}$$

$$\mu_1 h_1\left(\frac{\omega^2}{c_{s1}^2} - \beta^2\right) + \mu_2 h_2\left(\frac{\omega^2}{c_{s2}^2} - \beta^2\right) = \mu_e h_e\left(\frac{\omega^2}{c_{se}^2} - \beta^2\right) \tag{33}$$

In deriving the previous equations, h_1, h_2, and h_e are treated as small numbers [see the paragraph following Eq. (10)], and quantities such as h_1^2, h_2^2, and h_e^2 are neglected. Solving Eqs. (31–33) in association with Eqs. (26) and (27) yields

$$c_{se} = \left[\frac{\mu_e(h_1 + h_2)}{\mu_1 h_1/c_{s1}^2 + \mu_2 h_2/c_{s2}^2 - (\mu_e h_e - \mu_1 h_1 - \mu_2 h_2)\beta^2/\omega^2}\right]^{1/2} \tag{34}$$

$$\mu_e = \frac{h_1 + h_2}{h_1/\mu_1 + h_2/\mu_2} \tag{35}$$

and, by $c_s^2 = \mu/\rho$,

$$\rho_e = \frac{h_1\rho_1 + h_2\rho_2}{h_1 + h_2} - \frac{(\mu_e - \mu_1 h_1/h_e - \mu_2 h_2/h_e)\beta^2}{\omega^2} \tag{36}$$

Thus, it turns out that the effective shear modulus is a constant and equal to that determined from static problems. On the other hand, the effective density, together with the effective shear wave speed, is a variable depending on frequency ω as well as β, the parameter relating to the mode shape in the r direction. Therefore, ρ_e cannot be a material constant in the conventional sense except in the simple case $\beta = 0$ or $\mu_1 = \mu_2$. Despite the fact that it behaves rather singularly, this idealized quantity can be used to describe the natural vibration behavior of real composite laminates effectively and accurately.

Exact expression of the stationary value of natural frequency can be derived from Eqs. (30b) and (34) or Eq. (36):

$$\omega = \left[\frac{\mu_e m^2\pi^2 h_e + \beta^2(\mu_1 h_1 + \mu_2 h_2)}{\mu_1 h_1/c_{s1}^2 + \mu_2 h_2/c_{s2}^2}\right]^{1/2}, \quad m = 1, 2, 3, \ldots \tag{37}$$

The stationary vibration mode is given by Eq. (30a) with ω substituted from that indicated in Eq. (37).

It is not difficult to extend the preceding formulas to an N-ply basic unit. The result is

$$c_{se}(N) = \left[\frac{\mu_e(N)(h_1 + h_2 + \cdots + h_N)}{\mu_1 h_1/c_{s1}^2 + \mu_2 h_2/c_{s2}^2 + \cdots + \mu_N h_N/c_{SN}^2 - [\mu_e(N)h_e - \mu_1 h_1 - \mu_2 h_2 - \cdots - \mu_N h_N]\beta^2/\omega^2}\right]^{1/2} \tag{38}$$

$$\mu_e(N) = \frac{h_1 + h_2 + \cdots + h_N}{h_1/\mu_1 + h_2/\mu_2 + \cdots + h_N/\mu_N} \tag{39}$$

$$\rho_e(N) = \frac{h_1\rho_1 + h_2\rho_2 + \cdots + h_N\rho_N}{h_1 + h_2 + \cdots + h_N} - \frac{[\mu_e(N) - \mu_1 h_1/h_e - \mu_2 h_2/h_e - \cdots - \mu_N h_N/h_e]\beta^2}{\omega^2} \tag{40}$$

$$\omega = \left[\frac{\mu_e(N)m^2\pi^2 h_e + (\mu_1 h_1 + \mu_2 h_2 + \cdots + \mu_N h_N)\beta^2}{\mu_1 h_1/c_{s1}^2 + \mu_2 h_2/c_{s2}^2 + \cdots + \mu_N h_N/c_{sN}^2}\right]^{1/2} \quad m = 1, 2, 3, \ldots \tag{41}$$

where $h_e = h_1 + h_2 + \cdots + h_N$.

Table 2 Numerical values of first five natural frequency parameters ω_k/c_s for torsional vibration

	k[a]				
P[b]	1	2	3	4	5
2	8.8582	11.7665	15.7624	17.9340	20.7641
4	9.7122	11.5224	15.9192	20.8415	24.2562
6	10.4692	13.9286	18.6593	20.7936	24.1371
8	10.9646	12.3525	14.2723	16.9190	21.6004
10	11.2619	12.4502	14.3902	17.1292	20.2934
12	11.4433	12.5025	14.4462	17.2007	20.4316
14	11.5594	12.5338	14.4775	17.2353	20.4784
16	11.6373	12.5540	14.4969	17.2552	20.5013
18	11.6917	12.5677	14.5099	17.2694	20.5148
20	11.7311	12.5776	14.5190	17.2764	20.5233
22	11.7605	12.5848	14.5256	17.2826	20.5292
24	11.7832	12.5903	14.5307	17.2871	20.5335
26	11.8008	12.5946	14.5345	17.2905	20.5367
28	11.8148	12.5980	14.5375	17.2933	20.5392
30	11.8263	12.6008	14.5400	17.2954	20.5411
32	11.8355	12.6030	14.5420	17.2971	20.5426
34	11.8432	12.6049	14.5436	17.2985	20.5440
36	11.8498	12.6064	14.5449	17.2998	20.5449
38	11.8552	12.6077	14.5461	17.3008	20.5459
40	11.8600	12.6089	14.5471	17.3016	20.5466
42	11.8640	12.6099	14.5480	17.3023	20.5473
44	11.8676	12.6106	14.5487	17.3030	20.5478
46	11.8706	12.6114	14.5493	17.3035	20.5483
48	11.8733	12.6120	14.5499	17.3040	20.5486
50	11.8757	12.6126	14.5504	17.3044	20.5490
52	11.8778	12.6131	14.5509	17.3048	20.5493
54	11.8797	12.6135	14.5513	17.3051	20.5496
56	11.8814	12.6140	14.5516	17.3055	20.5499
58	11.8828	12.6142	14.5519	17.3058	20.5501
60	11.8842	12.6146	14.5522	17.3060	20.5503
SV[c]	11.9037	12.6192	14.5562	17.3093	20.5532

[a]Mode number. [b]Number of plies. [c]Stationary value.

In applying the preceding formulas to real composite laminates, they are valid or exact under the condition that the thickness of the basic unit is small or approaching zero. A composite cylinder consisting of a number of basic units of the same type have effective material quantities equal to those of the basic units.

A numerical example, in which $a = H = 1$, $h_1 = h_2 = h_3 = \cdots = h_p = H/P$, $d_1 = d_3 = \cdots = d_{p-1} = 2c_s$, $d_2 = d_4 = \cdots = d_p = c_s$, $\mu_1 = \mu_3 = \cdots = \mu_{p-1} = 3\mu/2$, $\mu_2 = \mu_4 = \cdots = \mu_p = \mu$, and $\beta = \beta_3 = 8.1475$, is shown in Table 2. Again it is seen that the stationary frequency and mode shape can be used to solve the problem effectively and accurately.

Antiplane Shear (Transverse) Vibration

In this section, we consider the antiplane shear vibration of a layered composite beam. The rectangular cross section of the beam is represented in Fig. 4. Assuming that the lateral surfaces of the beam are fixed, the only nonvanishing displacement in the problem, the displacement in the axial direction, can be expressed as follows:

$$W_i = \sin\frac{m\pi y}{b}(A_i \sin m_i x_i + B_i \cos m_i x_i), \quad i = 1, 2, \ldots, P \tag{42}$$

where $m = 1, 2, 3, \ldots$, b is the height of the cross section, and A_i and B_i should be determined from the boundary and interface conditions, and

$$\frac{m^2\pi^2}{b^2} + m_i^2 = \frac{\rho_i}{\mu_i}\omega^2 \tag{43}$$

This problem has been investigated in Ref. 5, where a general solution has been given, and numerical values of natural frequencies of a 20-ply composite beam with $\mu_2 = \mu_4 \cdots = \mu_{20} = \mu_0$, $\mu_1 = \mu_3 \cdots = \mu_{19} = f\mu_0$, $a_1 = a_2 = \cdots = a_{20} = b/20$, $\rho_1 = \rho_2 = \cdots \rho_{20} = \rho_0$, and $b = 2$ have been presented.

Table 3 Numerical values of the first five natural frequency parameters $\rho_0\omega^2/\mu_2\pi^2$ for antiplane shear vibration

	m/n	1	2	3	4	5
f[a] = 2	1	0.708[b]	1.707	3.370	5.689	8.652
		(0.708)[c]	(1.708)	(3.375)	(5.708)	(8.708)
	2	1.833	2.834[d]	4.499	6.822	9.792
		(1.833)	(2.833)	(4.500)	(6.833)	(9.833)
	3	3.702	4.706	6.375	8.703	11.687
		(3.708)	(4.708)	(6.375)	(8.708)	(11.708)
$f = 3$	1	0.875	1.999	3.863	6.452	9.737
		(0.875)	(2.000)	(3.875)	(6.500)	(9.875)
	2	2.372	3.500	5.373	7.975	11.280
		(2.375)	(3.500)	(5.375)	(8.000)	(11.375)
	3	4.852	5.988	7.874	10.497	13.833
		(4.875)	(6.000)	(7.875)	(10.500)	(13.875)
$f = 4$	1	1.167	2.415	4.479	7.329	10.910
		(1.167)	(2.416)	(4.500)	(7.417)	(11.167)
	2	3.402	4.663	6.749	9.633	13.265
		(3.461)	(4.667)	(6.750)	(9.667)	(13.416)
	3	7.078	8.358	10.479	13.417	17.130
		(7.167)	(8.416)	(10.500)	(13.417)	(17.167)

[a]Modulus ratio.
[b]Value of natural frequency parameter given in Ref. 5.
[c]Stationary value computed from Eq. (46).
[d]Probably should be 2.833.

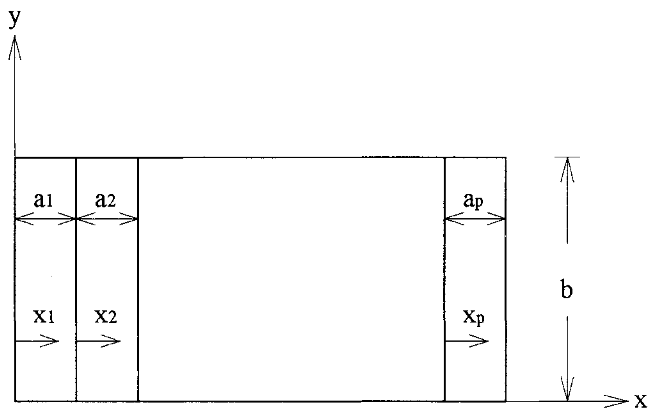

Fig. 4 Cross section of the layered beam.

Using the method developed in the preceding two sections, the effective modulus, effective density, and the stationary natural frequency of the 20-ply beam can be shown to be

$$\mu_e = \frac{2f\mu_2}{1+f} \tag{44}$$

$$\rho_e = \rho_0 - \frac{m^2\pi^2\mu_2}{2b^2\omega^2}\cdot\frac{(1-f)^2}{1+f} \tag{45}$$

$$\frac{\rho_0\omega^2}{\mu_2\pi^2} = \frac{2f}{1+f}\left(\frac{m^2+n^2}{b^2}\right) + \frac{m^2(1-f)^2}{2b^2(1+f)} \tag{46}$$

where $m, n = 1, 2, 3, \ldots$.

The stationary natural frequencies can easily be computed via Eq. (46), and the result is presented in Table 3 to compare with that given in Ref. 5. It is shown that the stationary values are always equal to or slightly larger than the natural frequency values of the 20-ply beam.[5] This fact again manifests the usefulness and validity of the stationary frequency concept.

It is of interest to note that in this case the density of each ply and the mean density of the beam are equal to ρ_0, yet the effective density still takes a different value.

Concluding Remarks

This article discusses the natural vibration behavior of composite laminates, with emphasis put on an idealized homogeneous system, whose natural frequency and vibration mode are equal to the stationary frequency and vibration mode of real composite laminates. This idealized homogeneous system can be used to describe the natural vibration behavior of real composite laminates effectively and accurately.

Utilizing the effective material quantities developed in this article, the natural vibration behavior of a composite laminate can simply and accurately be approximated by that of a homogeneous system. Moreover, the natural vibration behavior of the homogeneous system represents the ultimate state a composite laminate can attain in natural vibration.

The natural frequency approaches to a stationary value monotonically, and the natural vibration mode a stationary shape, as the ply number increases. The stacking sequence of different constituent plies takes effect only when the ply number is small, and for large ply numbers the effect diminishes asymptotically to zero.

The effective modulus of a composite laminate in natural vibration is a constant and identical to that found in static problems in all cases, whereas the effective density is a variable depending on frequency and other parameters and coincides with the mean density of the composite only in simple vibration modes.

These effective material quantities are developed in this article for some basic natural vibration problems. However, it is likely that they can straightforwardly be extended to more involved situations.

From Eqs. (20b) and (30b) in this paper, and also Eq. (53) in Ref. 5, where the main concern is to evaluate natural frequencies of composite laminates, it is seen that the natural frequency can straightforwardly be determined, once the effective wave speed, c_e or c_{se}, is known. By definition, wave speed depends on modulus and density. Consequently, for the mere purpose of determining the natural frequency, it can also be recognized that the effective wave speed, which varies with frequency and other parameters in general resulted from a varying effective modulus and a constant density.

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